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Existence of Best Approximations by Exponential Sums in Several Independent Variables

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In this paper we establish the existence of a best L_p approximation, $1 \leqslant p \leqslant \infty$, to a given function $f \in L_p(\mathscr{D})$, where $\mathscr{D} \subset \mathbf{R}^m$ is a bounded domain, from the family $V_n(S)$ of all nth order exponential sums in m independent variables for which the corresponding exponential parameters lie in the closed set $S \subseteq \mathbf{C}$. In so doing we extend the previously known existence theorem which corresponds to the special case where $m = 1$ and $\mathscr{D}$ is a finite interval.

1. INTRODUCTION

Let the complex valued function $y(\mathbf{t})$ be defined and have continuous partial derivatives of all orders throughout $\mathbf{R}^m$ and let $\mathscr{L}[y]$ denote the corresponding linear space (with complex scalars) which is generated by the functions

$$[D_1^{i_1} \cdots D_m^{i_m}]\, y(\mathbf{t}), \qquad i_1, \ldots, i_m = 0, 1, \ldots,$$

where

$$D_i = \partial/\partial t_i, \qquad i = 1, \ldots, m.$$

If $\mathscr{L}[y]$ has a finite dimension n we say that y is an exponential sum with order n. For example, the exponential sums

$$\begin{aligned} y(\mathbf{t}) &= 0, \\ y(\mathbf{t}) &= \exp(t_1 + t_2), \\ y(\mathbf{t}) &= t_1 \exp(t_1 + t_2), \\ y(\mathbf{t}) &= \exp(2t_1 + t_2) + \exp(t_1 + t_2) + \exp(t_1 + 2t_2) \end{aligned}$$

have orders 0, 1, 2, 3, respectively, in the case $m = 2$. The sum of two exponential sums having orders n_1, n_2 is again an exponential sum with order at most $n_1 + n_2$.

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For such an exponential sum y we know that $D_i{}^n y$ can be written as a linear combination of $y, D_i y, \ldots, D_i^{n-1} y$, $i = 1, \ldots, m$, so that y satisfies some system of partial differential equations of the form

$$[(D_i - \lambda_{i1}) \cdots (D_i - \lambda_{in})]\, y(\mathbf{t}) = 0, \qquad i = 1, \ldots, m. \tag{1}$$

Given a set $S \subseteq \mathbf{C}$ we define $V_n(S)$ to be the set of those exponential sums y with order at most n which satisfy some system (1) for which

$$\lambda_{ij} \in S, \qquad i = 1, \ldots, m, \quad j = 1, \ldots, n.$$

Our main objective in this note will be to establish the following m-dimensional generalization of Theorem 2 from [2].

THEOREM. *Let $\mathscr{D}$ be a bounded nonvoid open subset of $\mathbf{R}^m$, let $1 \leqslant p \leqslant \infty$, let S be a closed subset of $\mathbf{C}$, and let n be a positive integer. Then every $f \in L_p(\mathscr{D})$ has a best $\| \ \|_p$-approximation from $V_n(S)$.*

2. Three Basic Definitions

Essentially the same proof which is used for the special case where $m = 1$ and $\mathscr{D}$ is an interval can be used in the present situation provided that in addition to the above definition of $V_n(S)$ we formulate appropriate extensions of the parametrization $\mathscr{Y}_n(\mathbf{b}, \boldsymbol{\lambda})$, of the seminorms $\| \ \|_{p,\alpha}$, and of the concepts of U, V, W-sequences which are introduced in [2].

In general, any solution of the system (1) is an exponential sum of order at most n^m, and we may specify a particular solution of (1) by assigning the n^m initial conditions

$$[D_1^{i_1} \cdots D_m^{i_m}]\, y(\mathbf{0}) = b_{i_1 \cdots i_m}, \qquad 0 \leqslant i_1, \ldots, i_m \leqslant n - 1. \tag{2}$$

Indeed, by using a separation of variables technique we can easily construct a solution of the initial value problem (1), (2), and by using well-known methods (cf. [1, pp. 79, 80]) we can show that there is at most one such solution. We denote this unique solution by $\mathscr{Y}_n(\mathbf{b}, \boldsymbol{\lambda})$ where $\mathbf{b}$, $\boldsymbol{\lambda}$ represent the parameters $b_{i_1 \cdots i_m}$ and λ_{ij} appearing in (2) and (1), respectively. Thus each $y \in V_n(\mathbf{C})$ can be parametrized in the form $y = \mathscr{Y}_n(\mathbf{b}, \boldsymbol{\lambda})$ for some choice of $\mathbf{b}$, $\boldsymbol{\lambda}$ although this parametrization is not necessarily unique.

For each $\alpha \geqslant 0$ we define

$$\mathscr{D}_\alpha = \{\mathbf{t} \in \mathscr{D} : d(\mathbf{t}, \mathbf{R}^m \backslash \mathscr{D}) > \alpha\}, \tag{3}$$

where d is the usual Euclidean distance function, and we define the seminorm $\| \, \|_{p,\alpha}$ on $L_p(\mathscr{D})$ such that

$$\|f\|_{p,\alpha} = \|f \cdot \chi_\alpha\|_p ,$$

where χ_α is the characteristic function of $\mathscr{D}_\alpha$. Since $\mathscr{D}$ has nonvoid interior, we may select for later use some $\alpha_0 > 0$ such that $\mathscr{D}_{\alpha_0}$ also has nonvoid interior. {The choice $\alpha_0 = 1/3$ was used in [2] in the case where $m = 1$ and $\mathscr{D} = (0, 1)$.} The argument used to prove Lemma 1 in [2] shows that when S is bounded the seminorms $\| \, \|_{p,\alpha}$, $0 \leqslant \alpha \leqslant \alpha_0$, $1 \leqslant p \leqslant \infty$, are uniformly equivalent and the differential operators D_i, $i = 1,..., m$, are bounded on $V_n(S)$.

Given an exponential sum $y \in V_n(\mathbf{C})$ we define

$$\Lambda_i[y] = \bigcap \{\lambda_{i1},..., \lambda_{in}\}, \qquad i = 1,..., m,$$

with the intersections being taken over all possible choices of the exponential parameters λ_{ij} for which (1) holds. A sequence of exponential sums, $\{y_\nu\}$, from $V_n(\mathbf{C})$ will be called a U-sequence, a V-sequence, or a W-sequence as the corresponding spectral sets $\Lambda_i[y_\nu]$ satisfy the respective conditions

$$\max_i \liminf_\nu \{| \operatorname{Re} \lambda | : \lambda \in \Lambda_i[y_\nu]\} = +\infty,$$

$$\max_i \sup \left\{| \lambda | : \lambda \in \bigcup_{\nu=1}^{\infty} \Lambda_i[y_\nu]\right\} < +\infty,$$

or both of

$$\max_i \liminf_\nu \{| \operatorname{Im} \lambda | : \lambda \in \Lambda_i[y_\nu]\} = +\infty,$$

$$\max_i \sup \left\{| \operatorname{Re} \lambda | : \lambda \in \bigcup_{\nu=1}^{\infty} \Lambda_i[y_\nu]\right\} < +\infty.$$

From any sequence $\{y_\nu\}$ from $V_n(\mathbf{C})$ we can extract a subsequence (which we shall continue to call $\{y_\nu\}$) that may be decomposed in the form

$$y_\nu = u_\nu + v_\nu + w_\nu ,$$

where $\{u_\nu\}$, $\{v_\nu\}$, $\{w_\nu\}$ are U-, V-, W-sequences from $V_{n_1}(\mathbf{C})$, $V_{n_2}(\mathbf{C})$, $V_{n_3}(\mathbf{C})$, respectively, with $n_1 + n_2 + n_3 \leqslant n$.

3. Fundamental Properties of U-, V-, W-Sequences

The existence theorem which we wish to prove is an immediate corollary of the following lemma which gives four basic properties possessed by U-, V-, W-sequences (cf. the proof of Theorem 2 in [2].)

LEMMA. *Let* $\{u_\nu\}$, $\{v_\nu\}$, $\{w_\nu\}$ *be U-, V-, W-sequences, respectively, from* $V_n(\mathbf{C})$ *and let* $1 \leqslant p \leqslant \infty$.

(i) *If* $\{u_\nu + v_\nu + w_\nu\}$ *is a* $\| \|_p$*-bounded sequence from* $V_n(\mathbf{C})$, *then the component sequences* $\{u_\nu\}$, $\{v_\nu\}$, $\{w_\nu\}$ *are individually* $\| \|_p$*-bounded.*

(ii) *If* $\{v_\nu\}$ *is* $\| \|_p$*-bounded, then there is some subsequence of* $\{v_\nu\}$ *which uniformly converges on* $\bar{\mathscr{D}}$ *to some* $v \in V_n(\mathbf{C})$.

(iii) *If* $\{u_\nu\}$ *is* $\| \|_p$*-bounded, then* $\{u_\nu\}$ *converges uniformly to zero on every compact subset of* $\mathscr{D}$.

(iv) *If* $\{u_\nu + w_\nu\}$ *is a sequence from* $V_n(\mathbf{C})$ *and* $f \in L_p(\mathscr{D})$, *then*

$$\underline{\lim} \| f + u_\nu + w_\nu \|_p \geqslant \| f \|_p .$$

Proof. We have taken care to formulate the concepts given in the preceding sections in such a manner that the same arguments used in [2] may be used to establish the above lemma with (ii) being proved first as in Theorem 1 of [2] and with (i), (iii), and (iv) then being proved simultaneously using an induction on the order n as is done in Lemma 2 of [2]. This being the case we shall here supply only the critical argument used to show that (iii) holds in $V_n(\mathbf{C})$, $n \geqslant 1$, whenever (i) holds in $V_n(\mathbf{C})$ and (iii), (iv) hold in $V_{n-1}(\mathbf{C})$ leaving the remaining details to the reader.

As in [2] we need only consider the case where $p = 1$, where $\{u_\nu\}$ is a $\| \|_1$-normalized U-sequence from $V_n(\mathbf{C})$, and where sequences $\{\gamma_\nu\}$ from $\mathbf{C}$ and $\{\boldsymbol{\beta}_\nu\}$, $\{\boldsymbol{\delta}_\nu\}$ from $\mathbf{R}^m$ have been chosen so that if

$$\theta_\nu(\mathbf{t}) = \gamma_\nu \exp(\boldsymbol{\beta}_\nu \cdot \mathbf{t} + i\boldsymbol{\delta}_\nu \cdot \mathbf{t}), \qquad \mathbf{t} \in \mathbf{R}^m \tag{4}$$

(where the dot denotes the usual inner product in $\mathbf{R}^m$), then u_ν may be decomposed in such a manner that

$$\begin{aligned} u_\nu &= \theta_\nu[u_\nu{}^* + v_\nu{}^* + w_\nu{}^*] \\ &= \theta_\nu v_\nu{}^* + u_\nu^{**}, \end{aligned} \tag{5}$$

where $\{v_\nu{}^*\}$ is a convergent $\| \|_1$-normalized V-sequence which has limit v^*, where $\{u_\nu{}^*\}$ and $\{u_\nu^{**}\}$ are U-sequences from $V_{n-1}(\mathbf{C})$, where $\{w_\nu\}$ is a W-sequence, and where the order of u_ν is the sum of the orders of $u_\nu{}^*$, $v_\nu{}^*$, $w_\nu{}^*$ and also the sum of the orders of $v_\nu{}^*$, u_ν^{**}. Since $\{u_\nu\}$ is a U-sequence, we see from (4), (5) that

$$\lim | \boldsymbol{\beta}_\nu | = +\infty, \tag{6}$$

where we use $| \, |$ to denote the Euclidean norm in $\mathbf{R}^m$.

We select the sequence $\{\mathbf{t}_\nu\}$ from $\partial\mathscr{D}$, the (compact) boundary of $\mathscr{D}$, in such a manner that

$$\boldsymbol{\beta}_\nu \cdot \mathbf{t}_\nu = \max\{\boldsymbol{\beta}_\nu \cdot \mathbf{t} : \mathbf{t} \in \mathscr{D} \cup \partial\mathscr{D}\}, \qquad \nu = 1, 2,... \tag{7}$$

and (by passing to a subsequence, if necessary) assume that $\{\mathbf{t}_\nu\}$ has the limit $\mathbf{t}^* \in \partial\mathscr{D}$. Using (3), (7), and our previous choice of α_0, we also see that

$$\max\{\boldsymbol{\beta}_\nu \cdot \mathbf{t} : \mathbf{t} \in \mathscr{D}_\alpha \cup \partial\mathscr{D}_\alpha\} \leqslant \boldsymbol{\beta}_\nu \cdot \mathbf{t}_\nu - \alpha \mid \boldsymbol{\beta}_\nu \mid, \qquad 0 < \alpha \leqslant \alpha_0 . \tag{8}$$

Suppose now that α is chosen so that $0 < \alpha < \alpha_0/4$ and that K denotes the (nonvoid) intersection of $\mathscr{D}$ with the open ball in $\mathbf{R}^m$ which has center $\mathbf{t}^*$ and radius α. By dropping the first few terms of the sequence $\{\mathbf{t}_\nu\}$, if necessary, we may assume that $\mathbf{t}_\nu \in \partial K$ for each ν. We select the sequence $\{\boldsymbol{\tau}_\nu\}$ from ∂K in such a manner that

$$\boldsymbol{\beta}_\nu \cdot \boldsymbol{\tau}_\nu = \min\{\boldsymbol{\beta}_\nu \cdot \boldsymbol{\tau} : \boldsymbol{\tau} \in K \cup \partial K\}, \tag{9}$$

noting that

$$\mid \mathbf{t}_\nu - \boldsymbol{\tau}_\nu \mid \leqslant 2\alpha, \tag{10}$$

since both $\mathbf{t}_\nu$ and $\boldsymbol{\tau}_\nu$ are contained within a closed ball with radius α.

Using (4), (5), and (9) together with the inductive hypothesis that (iv) holds in $V_{n-1}(\mathbf{C})$ (with respect to the domain K), we find

$$\begin{aligned}
1 &= \lim \| u_\nu \|_1 \\
&\geqslant \overline{\lim} \int_K \mid u_\nu \mid \\
&\geqslant \overline{\lim} \left\{ \mid \theta_\nu(\boldsymbol{\tau}_\nu) \mid \int_K \mid u_\nu^* + v_\nu^* + w_\nu^* \mid \right\} \\
&\geqslant \overline{\lim} \mid \theta_\nu(\boldsymbol{\tau}_\nu) \mid \cdot \underline{\lim} \int_K \mid u_\nu^* + v^* + w_\nu^* \mid \\
&\geqslant \overline{\lim} \mid \theta_\nu(\boldsymbol{\tau}_\nu) \mid \cdot \int_K \mid v^* \mid .
\end{aligned}$$

And since the analytic function v^* (with $\| v^* \|_1 = 1$) cannot vanish identically on K, we infer the existence of some constant $B > 0$ such that

$$\mid \theta_\nu(\boldsymbol{\tau}_\nu) \mid \leqslant B, \qquad \nu = 1, 2,... .$$

Together with (4), (6), (8), and (10) this shows that

$$\begin{aligned}
\overline{\lim} \| \theta_\nu v_\nu^* \|_{\infty,3\alpha} &\leqslant \| v^* \|_\infty \overline{\lim} \max\{\mid \theta_\nu(\mathbf{t}) \mid : \mathbf{t} \in \partial\mathscr{D}_{3\alpha}\} \\
&= \| v^* \|_\infty \overline{\lim}[\mid \theta_\nu(\boldsymbol{\tau}_\nu) \mid \cdot \exp[\boldsymbol{\beta}_\nu \cdot (\mathbf{t}_\nu - \boldsymbol{\tau}_\nu)] \\
&\qquad \times \max\{\exp[\boldsymbol{\beta}_\nu \cdot (\mathbf{t} - \mathbf{t}_\nu)] : \mathbf{t} \in \partial\mathscr{D}_{3\alpha}\}] \\
&\leqslant \| v^* \|_\infty \cdot B \cdot \overline{\lim}[\exp(2\alpha \mid \boldsymbol{\beta}_\nu \mid) \exp(-3\alpha \mid \boldsymbol{\beta}_\nu \mid)] \\
&= 0.
\end{aligned} \tag{11}$$

Since $\{u_\nu\}$ is $\| \|_1$-normalized, we see from (5) and (11) that $\{u_\nu^{**}\}$ must be $\| \|_{1,3\alpha}$-bounded. Finally, by using (5), (11), and the inductive hypothesis that (iii) holds with respect to $\{u_\nu^{**}\}$ and the domain $\mathscr{D}_{3\alpha}$, we obtain

$$\overline{\lim} \| u_\nu \|_{\infty,4\alpha} \leqslant \overline{\lim} \| \theta_\nu v_\nu \|_{\infty,4\alpha} + \overline{\lim} \| u_\nu^{**} \|_{\infty,4\alpha} = 0$$

and from the arbitrariness of α we obtain (iii), thus completing the argument. ∎

4. Extensions to Vector Valued Exponential Sums

Each of the definitions given in the introduction can be used in the case where $\mathbf{y}: \mathbf{R}^m \to \mathbf{C}^r$ is r-vector valued for some fixed integer $r \geqslant 1$, and our previous definitions of U-, V-, W-sequences can also be used when we choose to work with r-vector valued exponential sums. The above lemma can be extended to apply to U-, V-, W-sequences of r-vector valued exponential sums by making use of the known special case where $r = 1$. The basic existence theorem stated in the introduction now applies when we approximate a given r-vector valued function $\mathbf{f} \in L_p(\mathscr{D})$ with the r-vector valued exponential sums from $V_n(S)$ with the proof being the same as that for the special case where $r = 1$.

References

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